Transverse Vibrations of a Trapezoidal Cantilever Plate of Variable Thickness

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Abstract

NATURAL frequencies of a trapezoidal cantilever plate of variable thickness, which model an aircraft wing structure approximately, are determined using energy techniques. Characteristic orthogonal polynomials in two variables are constructed to describe the structural deflections, which are used in the Rayleigh-Ritz method to obtain the natural frequencies. The fundamental natural frequencies for different parameter values are also obtained using a deflection function containing an optimized exponent. The first technique can be used to obtain the natural frequencies and corresponding mode shapes for the preliminary design of aircraft wing structures.

Contents

Introduction

For the determination of the first few natural frequencies, an aircraft wing can be modeled as a trapezoidal plate with variable thickness, as shown in Fig. 1. Bhat[1] proposed the use of beam characteristic orthogonal polynomials as deflection functions for the vibration of plates of rectangular geometry and characteristic orthogonal polynomials in two variables[2] for plates of arbitrary geometry. In the present paper, characteristic orthogonal polynomials in two variables are employed in the Rayleigh-Ritz technique to solve for the vibration of cantilever trapezoidal plates with variable thickness. The results are compared with those obtained using the Rayleigh method with an optimized exponent in the deflection expression.[3]

Characteristic Orthogonal Polynomials in Two Variables

A polynomial in two variables can be constructed by using the following sequence of monomials:

$$f_i(x,y) \rightarrow 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2y, \ldots,\ x^{n-k}\,y^k \tag{1}$$

The first member in the orthogonal polynomial set is constructed so as to satisfy at least the geometrical boundary conditions of the structure. If the plate has m number of geometrical boundary conditions, the first member can be assumed with $(m+1)$ terms in the form

$$\phi_1(x,y) = \sum_{i=0}^{m} c_i f_i(x,y) \tag{2}$$

Using the m boundary conditions in $\phi_1(x,y)$, all the constants c_i can be expressed in terms of one constant, say c_0, which can be arbitrarily set as unity. Arranging pairs of integers (n,k) $n \geq k \geq 0$, by lexicographic ordering, another pair of integers $(m,\ell)<(n,k)$ if either $m<n$ or $m=n$ and $\ell<k$. The $\phi_1(x,y)$ can be expressed as a power series in terms of the monomials in Eq. (1) as

$$\phi_1(x,y) = p_{n,k}(x,y) = \sum_{(m,\ell)\leq(n,k)} c(m,\ell;n,k)x^{m-\ell}y^{\ell} \tag{3}$$

The following recurrence relations are used to construct orthogonal polynomials[4]:

$$xp_{n,k}(x,y) = \sum_{(m,\ell)=(n-1,k)}^{(n+1,k)} a(m,\ell;n,k)p_{m,\ell}(x,y) \tag{4}$$

$$yp_{n,k}(x,y) = \sum_{(m,\ell)=(n-1,k-1)}^{(n+1,k+1)} b(m,\ell;n,k)p_{m,\ell}(x,y) \tag{5}$$

The constants $a(m,\ell;n,k)$ and $b(m,\ell;n,k)$ can be evaluated using the orthogonality relation

$$\iint \eta(x,y)p_{n,k}(x,y)\cdot p_{m,\ell}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \text{if} \quad (m,\ell)\neq(n,k)$$
$$= \mu_{n,k} \quad \text{if} \quad (m,\ell)\neq(n,k) \tag{6}$$

where the integration is over the plate area and $\eta(x,y)$ is a weight function. For example, $\phi_2(x,y)$ may be obtained using Eq. (4) as

$$x\phi_1(x,y) = a(n,k;n,k)\ \phi_1(x,y) + a(n+1,k;n,k)\ \phi_2(x,y) \tag{7}$$

The constant $a(n,k;n,k)$ can be obtained using Eq. (6) as

$$a(n,k;n,k) = \frac{\iint \eta(x,y)\cdot x\cdot\phi_1^2(x,y)\,\mathrm{d}x\,\mathrm{d}y}{\iint \eta(x,y)\ \phi_1^2(x,y)\,\mathrm{d}x\,\mathrm{d}y} \tag{8}$$

For plates of variable thickness, the weight function $\eta(x,y)$ can be chosen to describe the thickness variation.

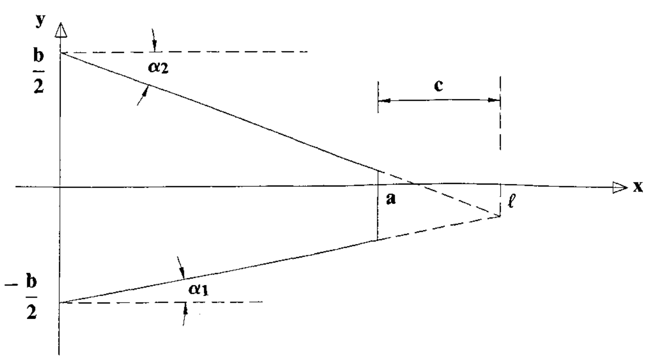

Fig. 1 Trapezoidal cantilever plate.

Received July 21, 1987; synoptic received March 10, 1988. Full paper available from National Technical Information Service, Springfield, VA 22151, at standard price (available upon request).

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The first member of the orthogonal set is taken as

$$\phi_1(x,y) = x^2[1 + m_0 h_0(x/a)] \tag{9}$$

where $m_0 = (a/h_0)\tan\alpha_0$, h_0 is plate thickness at $x = 0$ and a is shown in Fig. 1. Two more members are constructed using Eqs. (4-8) as

$$\phi_2(x,y) = x\phi_1 - a_1\phi_1 \tag{10}$$

$$\phi_3(x,y) = y\phi_1 - b_2\phi_2 - b_1\phi_1 \tag{11}$$

where

$$a_1 = \frac{\iint \eta(x,y)x\phi_1^2 \, dx \, dy}{\iint \eta(x,y)\phi_1^2 \, dx \, dy}$$

$$b_2 = \frac{\iint \eta(x,y)y\phi_1\phi_2 \, dx \, dy}{\iint \eta(x,y)\phi_2^2 \, dx \, dy}$$

$$b_1 = \frac{\iint \eta(x,y)y\phi_1^2 \, dx \, dy}{\iint \eta(x,y)\phi_1^2 \, dx \, dy}$$

The weight function for the structure in Fig. 1 is chosen as

$$\eta(x,y) = 1 + m_0 h_0 \, x/a \tag{12}$$

Eigenvalue Analysis

The plate deflection is expressed in the form

$$W(x,y) = \sum_{i=1}^{n} A_i\phi_i(x,y) \tag{13}$$

The kinetic and strain energies of the plate are

$$T_{\max} = \tfrac{1}{2}\rho\omega^2 \iint h(x,y)W^2(x,y) \, dx \, dy \tag{14}$$

$$U_{\max} = \tfrac{1}{2} \iint D[W_{xx}^2 + W_{yy}^2 + 2\nu W_{xx}W_{yy} + 2(1-\nu)W_{xy}^2] \, dx \, dy \tag{15}$$

where ρ is density of plate material, ν Poisson's ratio, $h(x,y)$ the variable thickness, $D = Eh^3(x,y)/12(1-\nu^2)$ the flexible rigidity, and subscripts x and y denote differentiation with respect to the subscript variables as many times as they appear. Substituting the deflection function into the energy expressions and minimizing the Rayleigh quotient with respect to coefficients A_i yields the eigenvalue problem. The solution of the eigenvalue problem will provide the natural frequencies and mode shapes.

Rayleigh Method with Optimized Exponent

A first-order approximation for the fundamental mode of vibration for the cantilever trapezoidal plate in Fig. 1 is obtained by disregarding the deflection variation in the y direction. The deflection is assumed as

$$W = A(x^\gamma + \beta_3 x^3 + \beta_2 x^2 + \beta_1 x + \beta_0) \tag{16}$$

where γ is an exponent that will be optimized later. The constants β_i, $i = 0,1,2,3$ are obtained by satisfying the boundary conditions in the x direction, $W(0) = W^{\text{I}}(0) = W^{\text{II}}(a) = W^{\text{III}}(a) = 0$. Substituting for W in the energy expressions in Eqs. (14) and (15), the fundamental natural frequency is obtained in terms of the exponent τ. Since ω_1 is an upper bound, it can be optimized with respect to the exponent γ as

$$\frac{\partial\omega_1}{\partial\gamma} = 0 \tag{17}$$

Table 1 Fundamental frequencies of cantilever trapezoidal plates, $\lambda_1 = \rho h_0 a^4\omega_1^2/D_0$

			m^0					
			−0.50		−0.25		0	
a/b	m_1	m_2	1)	2)	1)	2)	1)	2)
		−0.25	4.85	5.02	4.41	4.72	4.44	4.58
0.80	0.50	−0.50	5.72	5.89	5.52	5.55	5.40	5.39
		−0.625	6.54	6.58	6.22	6.23	6.02	6.07
		−0.25	4.49	4.60	4.24	4.33	4.13	4.20
0.60	0.50	−0.50	4.90	5.02	4.70	4.72	4.58	4.58
		−0.625	4.14	5.29	5.00	4.98	4.81	4.83
		−0.25	4.22	4.29	4.01	4.04	3.89	3.91
0.40	0.50	−0.50	4.41	4.49	4.22	4.22	4.10	4.09
		−0.625	4.51	4.60	4.30	4.33	4.19	4.20
		−0.25	5.12	5.39	4.38	5.08	4.61	4.93
0.80	0.625							
		−0.625	7.51	7.66	7.30	7.32	7.16	7.15
		−0.25	4.64	4.79	4.31	4.51	4.21	4.37
0.60	0.625							
		−0.625	5.48	5.62	5.27	5.29	5.15	5.14
		−0.25	4.29	4.39	4.06	4.13	3.94	4.01
0.40	0.625							
		−0.625	4.63	4.73	4.44	4.44	4.32	4.31

1) Characteristic orthogonal polynomials in two variables.
2) Rayleigh method with optimized exponent.

Numerical Results

Table 1 presents a comparison of frequency coefficients obtained by the two approaches previously described where $\lambda_1 = (\rho h_0 a^4/D_0)\omega_1^2$ and $D_0 = Eh_0^3/[12(1-\gamma^2)]$. Since only three terms are used in the first method using orthogonal polynomials, only the fundamental frequency values are listed. In general, the agreement is good, from a practical engineering viewpoint. Rayleigh results with optimized exponent provide good approximation for the fundamental frequency. Accuracy of the natural frequencies can be increased by considering a greater number of orthogonal polynomial terms in the deflection expression in Eq. (13).

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